SPECTRA OF THE TRANSLATIONS AND WIENER-HOPF OPERATORS ON $L^2_\omega(\mathbb{R}^+)$

VIOLETA PETKOVA

ABSTRACT. We study bounded operators T on the weighted space $L^2_\omega(\mathbb{R}^+)$ commuting either with the "right shift operators" $(R_t)_{t\geq 0}$, or "left shift operators" $(L_{-t})_{t\geq 0}$, and we establish the existence of a symbol μ of T. We characterize completely the spectrum $\sigma(R_t)$ of the operator R_t proving that

$$\sigma(R_t) = \{z \in \mathbb{C} : |z| \leq e^{t\alpha_0}\},$$

where α_0 is the growth bound of $(R_t)_{t\geq 0}$. We obtain a similar result for the spectrum of (L_{-t}), $t > 0$. Moreover, for a bounded operator T commuting with R_t, $t \geq 0$, we establish the inclusion $\overline{\mu(\mathcal{O})} \subset \sigma(T)$, where $\mathcal{O} = \{z \in \mathbb{C} : \operatorname{Im} z < \alpha_0\}$.

Key Words: translations, spectrum of Wiener-Hopf operator, semigroup of translations, weighted spaces, symbol

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1. INTRODUCTION

Let ω be a weight on $\mathbb{R}^+$. It means that ω is a positive, not vanishing, continuous function on $\mathbb{R}^+$ such that

$$0 < \inf_{x\geq 0} \frac{\omega(x+t)}{\omega(x)} \leq \sup_{x\geq 0} \frac{\omega(x+t)}{\omega(x)} < +\infty, \forall t \in \mathbb{R}^+.$$

For example e^x and e^{-x} are weights on $\mathbb{R}^+$ and we will see later that

$$\sup_{x\geq 0} \frac{\omega(x+t)}{\omega(x)} \leq Ce^{mt},$$

where C and m are constants.
Let $L^2_\omega(\mathbb{R}^+)$ be the set of measurable functions on $\mathbb{R}^+$ such that

$$\int_0^\infty |f(x)|^2 \omega(x)^2 dx < +\infty.$$

The space $H = L^2_\omega(\mathbb{R}^+)$ equipped with the scalar product

$$< f, g >= \int_{\mathbb{R}^+} f(x)\overline{g}(x)\omega(x)^2 dx,\ f \in L^2_\omega(\mathbb{R}^+),\ g \in L^2_\omega(\mathbb{R}^+)$$

Universite de Lorraine, LMAM, UMR 7122, Bat A, Ile de Saulcy, 57045 Metz Cedex 1, France, petkova@univ-metz.fr.

and the related norm $\|.\|$ is a Hilbert space. Let $C_c^\infty(\mathbb{R})$ (resp. $C_c^\infty(\mathbb{R}^+)$) be the space of C^∞ functions on $\mathbb{R}$ (resp. $\mathbb{R}^+$) with compact support in $\mathbb{R}$ (resp. $\mathbb{R}^+$). Notice that $C_c^\infty(\mathbb{R}^+)$ is dense in $L^2_\omega(\mathbb{R}^+)$. Let r_+ be the restriction from $L^1_{loc}(\mathbb{R}^-) \oplus L^2_\omega(\mathbb{R}^+)$ into $L^2_\omega(\mathbb{R}^+)$ and let $\mathbf{r}_+$ be the restriction from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^+)$. Let ℓ_0 be the extension of a function defined on $\mathbb{R}^+$ into a function on $\mathbb{R}$ setting $\ell_0 f(x) = 0, \forall x \in \mathbb{R}^-$. These notations are compatible with those used for example in [5].

For $t \in \mathbb{R}$ and a function defined on $\mathbb{R}$ we denote $(S_t f)(x) = f(x-t)$. Then for $t > 0$ we introduce the "right shift" operator on $L^2_\omega(\mathbb{R}^+)$ by $R_t := r_+ S_t \ell_0$. For simplicity R_1 will be denoted by R. For $t > 0$ we define the "left shift" operator on $L^2_\omega(\mathbb{R}^+)$ by $L_{-t} f := r_+ S_{-t} \ell_0 f$. It is clear that R_t and L_{-t} map $L^2_\omega(\mathbb{R}^+)$ into $L^2_\omega(\mathbb{R}^+)$, while the shift S_t acts on functions defined on $\mathbb{R}$. The notations R_t and L_{-t} are similar to those for the operators on the spaces of sequences (see [16]). Let I be the identity operator on $L^2_\omega(\mathbb{R}^+)$.

Definition 1. *A bounded operator T on $L^2_\omega(\mathbb{R}^+)$ is called a Wiener-Hopf operator if*

$$L_{-t} T R_t f = Tf, \ \forall t \in \mathbb{R}^+, \ f \in L^2_\omega(\mathbb{R}^+).$$

More general Wiener-Hopf operators have been intensively studied in the literature (see [18] and the references given there). There exist also many results about Toeplitz operators on weighted Hardy spaces (see [3], [2]). Such operators have some similarities with Wiener-Hopf operators.

Every Wiener-Hopf operator T has a representation by a convolution. The reader may find a proof in [12] where the arguments of [9], [10] are used. More precisely, there exists a distribution $\mu \in \mathcal{D}'(\mathbb{R})$ such that

$$Tf(x) = (\mu * \ell_0 f)(x), \ \forall f \in C_c^\infty(\mathbb{R}^+), \ x \in \mathbb{R}^+.$$

If $\phi \in C_c^\infty(\mathbb{R})$ then the operator

$$L^2_\omega(\mathbb{R}^+) \ni f \longrightarrow r_+(\phi * \ell_0 f)$$

is a Wiener-Hopf operator and we will denote it by T_ϕ.

A bounded operator T on $L^2_\omega(\mathbb{R}^+)$ commuting either with $R_t, \forall t > 0$, or with $L_{-t}, \forall t > 0$ is a Wiener-Hopf operator. On the other hand, every operator $\alpha L_{-t} + \beta R_t$ with $t > 0$, α, $\beta \in \mathbb{C}$ is a Wiener-Hopf operator. It is clear that the set of Wiener-Hopf operators is not a sub-algebra of the algebra of the bounded operators on $L^2_\omega(\mathbb{R}^+)$.

Notice also that

$$(L_{-t} R_t) f = f, \ \forall f \in L^2_\omega(\mathbb{R}^+), \ t > 0,$$

but it is obvious that $(R_t L_{-t}) f \neq f$, for all $f \in L^2_\omega(\mathbb{R}^+)$ with a support not included in $]t, +\infty[$. The fact that R_t is not invertible leads to many difficulties in contrast to the case when we deal with the space $L^2_\omega(\mathbb{R})$. The later space has been considered in [14] and [15] and the author has studied the operators commuting with the translations on $L^2_\omega(\mathbb{R})$ characterizing their spectrum. The group of translations on $L^2_\omega(\mathbb{R})$ is commutative and the investigation of its spectrum is easier. In this work, first we apply some ideas used in [14] and [15] to study Wiener-Hopf operators on $L^2_\omega(\mathbb{R}^+)$. For this purpose it

is necessary to treat two semigroups of not invertible operators instead of a group of invertible operators. More precisely, we must deal with the semigroups $(R_t)_{t\geq 0}$ and $(L_{-t})_{t\geq 0}$ on $L^2_\omega(\mathbb{R}^+)$. For our analysis it is more convenient to replace the weight ω by another one. To do this, given a weight ω, denote by $\overline{\omega}$ the function $\overline{\omega}(t) = \sup_{x\geq 0} \frac{\omega(x+t)}{\omega(x)}$. Define $\omega_0(x) = \exp\Big(\int_1^2 \ln(\omega(x+t))dt\Big)$. Following [1], ω_0 is a well defined weight on $\mathbb{R}^+$ and, moreover, we have

$$\overline{\omega_0}(t) \leq e^{mt},\ \forall t \geq 0, \tag{1.1}$$

where m is a constant. The weights ω and ω_0 are equivalent (see [1]), that is there exist constants C_1 and C_2 such that

$$C_1\omega_0(x) \leq \omega(x) \leq C_2\omega_0(x),\ \forall x \in \mathbb{R}^+.$$

Taking into account that $\|R_t\| = \overline{\omega}(t)$, from (1.1) we get the estimate $\|R_t\| \leq Ce^{mt}$, $\forall t \in \mathbb{R}^+$. A similar estimate holds for the semigroup $(L_{-t})_{t\geq 0}$.

Denote by $\rho(B)$ (resp. $\sigma(B)$) the spectral radius (resp. the spectrum) of an operator B. Introduce the ground orders of the semigroups $(R_t)_{t\geq 0}$ and $(L_{-t})_{t\geq 0}$ by

$$\alpha_0 = \lim_{t\to\infty} \frac{1}{t} \ln \|R_t\|,\ \alpha_1 = \lim_{t\to\infty} \frac{1}{t} \ln \|L_{-t}\|.$$

Then it is well known (see for example [4]) that we have

$$\rho(R_t) = e^{\alpha_0 t},\ \rho(L_{-t}) = e^{\alpha_1 t}.$$

Let $\mathcal{I}$ by the interval $[-\alpha_1, \alpha_0]$ and define

$$\Omega := \Big\{z \in \mathbb{C} :\ e^{-\alpha_1} \leq |z| \leq e^{\alpha_0}\Big\}.$$

Notice that $\alpha_1 + \alpha_0 \geq 0$. Indeed, for every $n \in \mathbb{N}$ we have $L_{-n}R_n = I$ and

$$1 \leq \limsup_{n\to\infty} \|(L_{-1})^n\|^{1/n} \limsup_{n\to\infty} \|R^n\|^{1/n} = e^{\alpha_1}e^{\alpha_0}$$

which yields the result. For a function f and $a \in \mathbb{C}$ we denote by $(f)_a$ the function

$$(f)_a : x \longrightarrow f(x)e^{ax}.$$

Denote by $\mathcal{F}$ the usual Fourier transformation on $L^2(\mathbb{R})$ and set $\hat{f} = \mathcal{F}f$, for $f \in L^2(\mathbb{R})$. For a function $f \in L^2(\mathbb{R}^+)$, we define $\tilde{f}$ as the Fourier transform of the function f extended by 0 on $\mathbb{R}^-$.

Our first result is the following

Theorem 1. *Let $a \in \mathcal{I} = [-\alpha_1,\ \alpha_0]$ and let T be a Wiener-Hopf operator. There exists $h_a \in L^\infty(\mathbb{R})$ such that for every $f \in L^2_\omega(\mathbb{R}^+)$ satisfying $(f)_a \in L^2(\mathbb{R}^+)$,*

$$(Tf)_a = \mathbf{r}_+\mathcal{F}^{-1}(h_a\widetilde{(f)_a}) \tag{1.2}$$

and

$$\|h_a\|_\infty \leq C\|T\|,$$

where C is a constant independent of a. Moreover, if $\alpha_1+\alpha_0>0$, the function h defined on $U=\{z\in\mathbb{C}:\ \mathrm{Im}\, z\in[-\alpha_1,\ \alpha_0]\}$ by $h(z)=h_{\mathrm{Im}\, z}(\mathrm{Re}\, z)$ is holomorphic on $\overset{\circ}{U}$.

Definition 2. *The function h defined in Theorem 1 is called the symbol of T.*

A weaker result that Theorem 1 has been proved in [12] where the representation (1.2) has been obtained only for functions $f\in C_c^\infty(\mathbb{R}^+)$ which is too restrictive for the applications to the spectral problems studied in Section 3 and Section 4. On the other hand, in the proof in [12] there is a gap in the approximation argument. Indeed in the proof of Lemma 2 in [12], there is one argument from Lemma 6 in [11] which can be applied only if the function $\mathbb{R}^+\ni t\longrightarrow R_t$ is continuous with respect to the operator norm topology on the set of bounded operators on $L^2_\omega(\mathbb{R}^+)$. Guided by the approach in [15], in this work we prove a stronger version of the result of [12] applying other techniques based essentially on the spectral theory of semigroups. On the other hand, in many interesting cases as $\omega(x)=e^x$, $\omega(x)=e^{-x}$, we have $\alpha_0+\alpha_1=0$ and the result of Theorem 1 is not satisfied since the symbol of T is defined only on the line $\mathrm{Im}\, z=\alpha_0$. To obtain more complete results we introduce the following class of operators.

Definition 3. *Denote by $\mathcal{M}$ (resp. $\mathcal{W}$) the set of bounded operators on $L^2_\omega(\mathbb{R}^+)$ commuting with R_t, $\forall t>0$ (resp. L_{-t}, $\forall t>0$).*

For operators in $\mathcal{M}$ or $\mathcal{W}$ we obtain a stronger version of Theorem 1.

Theorem 2. *Let T be a bounded operator commuting with $(R_t)_{t>0}$ (resp. $(L_{-t})_{t>0}$). Let $a\in J=]0,\alpha_0]$ (resp. $K=]0,\alpha_1]$). There exists $h_a\in L^\infty(\mathbb{R})$ such that for every $f\in L^2_\omega(\mathbb{R}^+)$ satisfying $(f)_a\in L^2(\mathbb{R}^+)$, we have*

$$(Tf)_a=\mathbf{r}_+\mathcal{F}^{-1}(h_a\widetilde{(f)_a})$$

and

$$\|h_a\|_\infty\leq C\|T\|,$$

where C is a constant independent of a. Moreover, the function h defined on $\mathcal{O}=\{z\in\mathbb{C}:\ \mathrm{Im}\, z<\alpha_0\}$ (resp. $\mathcal{V}=\{z\in\mathbb{C}:\ \mathrm{Im}\, z>-\alpha_1\}$) by $h(z)=h_{\mathrm{Im}\, z}(\mathrm{Re}\, z)$ is holomorphic on $\mathcal{O}$ (resp. $\mathcal{V}$).

Our main spectral result is the following

Theorem 3. *We have*

$$(i)\quad \sigma(R_t)=\{z\in\mathbb{C},\ |z|\leq e^{\alpha_0 t}\},\ \forall t>0. \tag{1.3}$$

$$(ii)\quad \sigma(L_{-t})=\{z\in\mathbb{C},\ |z|\leq e^{\alpha_1 t}\},\ \forall t>0. \tag{1.4}$$

iii) Let $T\in\mathcal{M}$ and let μ_T be the symbol of T. Then we have

$$\overline{\mu_T(\mathcal{O})}\subset\sigma(T). \tag{1.5}$$

iv) Let $T \in \mathcal{W}$ and let μ_T be the symbol of T. Then we have

$$\overline{\mu_T(\mathcal{V})} \subset \sigma(T). \tag{1.6}$$

It is important to note that for $T \in \mathcal{M}$ (resp. $\mathcal{W}$) and $\lambda \in \mathbb{C}$, if the resolvent $(T-\lambda I)^{-1}$ exists, then $(T-\lambda I)^{-1}$ is also in $\mathcal{M}$ (resp. $\mathcal{W}$). In general, if T is a Wiener-Hopf operator and $\lambda \in \mathbb{C}$, even if the resolvent $(T-\lambda I)^{-1}$ exists, $(T-\lambda I)^{-1}$ could be not a Wiener-Hopf operator. For more information about Wiener-Hopf operators the reader may consult [5] and [7].

The result in Theorem 3 cannot be obtained from a spectral calculus which is unknown and quite difficult to construct for the operators in $\mathcal{M}$ or $\mathcal{W}$. On the other hand, our analysis shows the importance of the existence of symbols and this was our main motivation to establish Theorem 1 and Theorem 2.
The spectrum of the weighted right and left shifts on $l^2(\mathbb{R}^+)$, denoted respectively by $\mathbf{R}$ and $\mathbf{L}$, has been studied in [16]. In particular, it was shown that

$$\sigma(\mathbf{R}) = \sigma(\mathbf{L}) = \{z \in \mathbb{C},\ |z| \leq \rho(\mathbf{R})\}. \tag{1.7}$$

In this special case the operators $\mathbf{R}$ and $\mathbf{L}$ are adjoint, while this property in general is not true for R and L_{-1}.

The equalities (1.3), (1.4) are the analogue of (1.7) in $L^2_\omega(\mathbb{R}^+)$ however our proof is quite different from that in [16] and we use essentially Theorem 2. Moreover, these results agree with the spectrum of composition operator studied in [17] and the circular symmetry about 0. In the standard case $\omega = 1$, the spectral results (1.3), (1.4) are well known (see, for example Chapter V, [4]). Their proof in this special case is based on the fact that the spectrum of the generator A of $(R_t)_{t\geq 0}$ is in $\{z \in \mathbb{C},\ \mathrm{Re}\, z \leq 0\}$ and the spectral mapping theorem for semigroups yields $\sigma(R_t) = \{z \in \mathbb{C},\ |z| \leq 1\}$. Notice also that in this case we have

$$s(A) = \sup\{\mathrm{Re}\,\lambda :\ \lambda \in \sigma(A)\} = \alpha_0 = 0,$$

so the spectral bound $s(A)$ of A is equal to the ground order and there is no *spectral gap*. In the general setting we deal with it is quite difficult to describe the spectrum of A. Consequently, we cannot obtain (1.3) from the spectrum of A and our techniques are not based on $\sigma(A)$. Moreover, if for the semigroup $(R_t)_{t\geq 0}$ on $L^2_\omega(\mathbb{R}^+)$ we can apply the spectral mapping theorem, since R_t preserves positive functions (see [19], [20]), in general this is not true for other Hilbert spaces of functions and we could have a spectral gap $s(A) < \alpha_0$. This shows the importance of our approach which works also for more general Hilbert spaces H of functions (see the conditions on H listed below). To our best knowledge it seems that Theorem 3 is the first result in the literature giving a complete characterization of $\sigma(R_t)$ and $\sigma(L_{-t})$ on the spaces $L^2_\omega(\mathbb{R}^+)$. On the other hand, for the weighted two-sided shift $\mathbf{S}$ in $L^2_\omega(\mathbb{R})$ a similar result has been established in [15] saying that

$$\sigma(\mathbf{S}) = \{z \in \mathbb{C} :\ \frac{1}{\rho(\mathbf{S}^{-1})} \leq |z| \leq \rho(\mathbf{S})\}.$$

Following the arguments in [14], the results of this paper may be extended to a larger setup. Indeed, instead of $L^2_\omega(\mathbb{R}^+)$ we may consider a Hilbert space H of functions on $\mathbb{R}^+$ satisfying the following conditions:

(H1) $C_c(\mathbb{R}^+) \subset H \subset L^1_{loc}(\mathbb{R}^+)$, with continuous inclusions, and $C_c(\mathbb{R}^+)$ is dense in H.

(H2) For every $t \in \mathbb{R}$, we have $R_t(H) \subset H$ (resp. $L_{-t}(H) \subset H$) and $\sup_{t\in K} \|R_t\| < +\infty$ (resp. $\sup_{t\in K} \|L_{-t}\| < +\infty$), for every compact set $K \subset \mathbb{R}^+$.

(H3) For every $\alpha \in \mathbb{R}$, let E_α be the operator defined by

$$E_\alpha : H \ni f \longrightarrow \Big(\mathbb{R} \ni x \longrightarrow f(x)e^{i\alpha x}\Big).$$

We have $E_\alpha(H) \subset H$ and moreover, $\sup_{\alpha\in\mathbb{R}} \|E_\alpha\| < +\infty$.

(H4) There exist $C_1 > 0$ and $a_1 \geq 0$ such that $\|R_t\| \leq C_1 e^{a_1|t|}$, $\forall t \in \mathbb{R}^+$.

(H5) There exist $C_2 > 0$ and $a_2 \geq 0$ such that $\|L_{-t}\| \leq C_2 e^{a_2|t|}$, $\forall t \in \mathbb{R}^+$.

Taking into account (H3), without lost of generality we may consider that in H we have $\|fe^{i\alpha\cdot}\| = \|f\|$. For the simplicity of the exposition we deal with the case $H = L^2_\omega(\mathbb{R}^+)$ and the reader may consult [14] for the changes necessary to cover the more general setup.

2. Proof of Theorem 1

Denote by A the generator of the semi-group $(R_t)_{t\geq 0}$. By using the arguments based on the spectral results for semigroups (see [6], [8]) we will prove the following

Lemma 1. *Let λ be such that $e^\lambda \in \sigma(R)$ and $\operatorname{Re}\lambda = \alpha_0$. Then there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of integers and a sequence $(f_{m_k})_{k\in\mathbb{N}}$ of functions of H such that*

$$\lim_{k\to\infty} \|\Big(e^{tA} - e^{(\lambda+2\pi i n_k)t}\Big) f_{m_k}\| = 0,\ \forall t \in \mathbb{R}^+,\ \ \|f_{m_k}\| = 1,\ \forall k \in \mathbb{N}. \tag{2.1}$$

Proof. We have to deal with two cases: (i) $\lambda \in \sigma(A)$, (ii) $\lambda \notin \sigma(A)$. In the case (i) λ is in the approximative point spectrum of A. This follows from the fact that for any $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \alpha_0$ we have $\mu \notin \sigma(A)$, since $s(A) \leq \alpha_0$.

Let μ_m be a sequence such that $\mu_m \to \lambda$, $\operatorname{Re}\mu_m > \alpha_0$. Then $\|(\mu_m I - A)^{-1}\| \geq (\text{dist}\,(\mu_m, spec(A)))^{-1}$, hence $\|(\mu_m I - A)^{-1}\| \to \infty$. Applying the uniform boundedness principle and passing to a subsequence μ_{m_k}, we may find $f \in H$ such that

$$\lim_{k\to\infty} \|(\mu_{m_k} I - A)^{-1} f\| = \infty.$$

Introduce $f_{m_k} \in D(A)$ defined by

$$f_{m_k} = \frac{(\mu_{m_k} I - A)^{-1} f}{\|(\mu_{m_k} I - A)^{-1} f\|}.$$

The identity

$$(\lambda - A)f_{m_k} = (\lambda - \mu_{m_k})f_{m_k} + (\mu_{m_k} - A)f_{m_k}$$

implies that $(\lambda - A)f_{m_k} \to 0$ as $k \to \infty$. Then the equality

$$(e^{tA} - e^{t\lambda})f_{m_k} = \Big(\int_0^t e^{\lambda(t-s)}e^{As}ds\Big)(A - \lambda)f_{m_k}$$

yields (2.1), where we take $n_k = 0$.
To deal with the case (ii), we repeat the argument in [14] and for the sake of completeness we present the details. We have $e^\lambda \in \sigma(e^A) \setminus e^{\sigma(A)}$. Applying the results for the spectrum of a semigroup in Hilbert space (see [6], [8]), we conclude that there exists a sequence of integers (n_k) such that $|n_k| \to \infty$ and

$$\|(A - (\lambda + 2\pi i n_k)I)^{-1}\| \geq k,\ \forall k \in \mathbb{N}.$$

We choose a sequence $(g_{m_k}) \in H$, $\|g_{m_k}\| = 1$ so that

$$\|(A - (\lambda + 2\pi i n_k)I)^{-1}g_{m_k}\| \geq k/2,\ \forall k \in \mathbb{N}$$

and define

$$f_{m_k} = \frac{(A - (\lambda + 2\pi i n_k)I)^{-1}g_{m_k}}{\|(A - (\lambda + 2\pi i n_k)I)^{-1}g_{m_k}\|}.$$

Next we have

$$(e^{tA} - e^{(\lambda+2\pi i n_k)t})f_{m_k} = \Big(\int_0^t e^{(\lambda+2\pi i n_k)(t-s)}e^{sA}ds\Big)(A - (2\pi i n_k + \lambda)I)f_{m_k}$$

and we deduce (2.1). □

Lemma 2. *Let λ be such that $e^\lambda \in \sigma(R)$ and $\operatorname{Re}\lambda = \alpha_0$. Then, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of integers and a sequence $(f_{m_k})_{k\in\mathbb{N}}$ of functions of H such that for all $t \in \mathbb{R}$,*

$$\lim_{k\to\infty}\Big\|\Big(r_+S_t\ell_0 - e^{(\lambda+2\pi i n_k)t}\Big)f_{m_k}\Big\| = 0,\ \ \|f_{m_k}\| = 1,\ \forall k \in \mathbb{N}. \tag{2.2}$$

Proof. Clearly, for $t \geq 0$ we get (2.2) by (2.1). Moreover, we have

$$\|(L_{-t} - e^{-(\lambda+2\pi i n_k)t})f_{m_k}\| = \|(L_{-t} - e^{-(\lambda+2\pi i n_k)t}L_{-t}R_t)f_{m_k}\|$$
$$\leq \|L_{-t}\||e^{-(\lambda+2\pi i n_k)t}|\Big\|\Big(e^{(\lambda+2\pi i n_k)t} - R_t\Big)f_{m_k}\Big\|,\ \forall t \in \mathbb{R}^+.$$

Thus

$$\lim_{k\to\infty}\|(L_{-t} - e^{-(\lambda+2\pi i n_k)t})f_{m_k}\| = 0$$

and this completes the proof of (2.2). □
Recall that for $\phi \in C_c^\infty(\mathbb{R})$, T_ϕ is the operator on $L^2_\omega(\mathbb{R}^+)$ given by

$$T_\phi(f) = r_+(\phi * \ell_0 f),\ \forall f \in L^2_\omega(\mathbb{R}^+).$$

Lemma 3. *For all $\phi \in C_c^\infty(\mathbb{R})$ and λ such that $e^\lambda \in \sigma(R)$ with $\operatorname{Re}\lambda = \alpha_0$ we have*

$$|\hat{\phi}(i\lambda + a)| \leq \|T_\phi\|,\ \forall a \in \mathbb{R}. \tag{2.3}$$

Proof. Let $\lambda \in \mathbb{C}$ be such that $e^{\lambda} \in \sigma(R)$ with $\operatorname{Re}\lambda = \alpha_0$ and let $(f_{m_k})_{k\in\mathbb{N}}$ be the sequence satisfying (2.2). Fix $\phi \in C_c^\infty(\mathbb{R})$ and consider

$$|\hat{\phi}(i\lambda + a)| = |\int_{\mathbb{R}} \langle \phi(t) e^{(\lambda - ia)t} f_{m_k}, f_{m_k}\rangle dt|$$

$$\leq \Big| \int_{\mathbb{R}} \Big\langle \phi(t) \Big(e^{(\lambda + i2\pi n_k)t} - r_+ S_t \ell_0 \Big) e^{-i(a+2\pi n_k)t} f_{m_k}, f_{m_k} \Big\rangle dt \Big|$$

$$+ \Big| \int_{\mathbb{R}} \langle \phi(t) r_+ S_t \ell_0 e^{-i(a+2\pi n_k)t} f_{m_k}, f_{m_k}\rangle dt \Big|.$$

The first term on the right side of the last inequality goes to 0 as $k \to \infty$ since by Lemma 1, for every fixed t we have

$$\lim_{k\to+\infty} \big\| e^{-i(a+2\pi n_k)t} \Big(e^{(\lambda + i2\pi n_k)t} - r_+ S_t \ell_0 \Big) f_{m_k} \big\| = 0.$$

On the other hand,

$$I_k = \Big| \int_{\mathbb{R}} < \phi(t) r_+ S_t \ell_0 e^{-i(a+2\pi n_k)t} f_{m_k}, f_{m_k} > dt \Big|$$

$$= \Big| \langle r_+ \Big[\int_{\mathbb{R}} \phi(t) e^{-i(a+2\pi n_k)t} \ell_0 f_{m_k}(. - t) dt \Big], f_{m_k}(.) \rangle \Big|$$

$$= \Big| \langle r_+ \int_{\mathbb{R}} \phi(. - y) e^{i(a+2\pi n_k)y} \ell_0 f_{m_k}(y) dy, e^{i(a+2\pi n_k)\cdot} f_{m_k}(.) \rangle \Big|$$

$$= \Big| \langle \Big(T_\phi (e^{i(a+2\pi n_k)\cdot} f_{m_k}) \Big), e^{i(a+2\pi n_k)\cdot} f_{m_k}(.) \rangle \Big|$$

and $I_k \leq \|T_\phi\|$. Consequently, we deduce that $|\hat{\phi}(i\lambda + a)| \leq \|T_\phi\|$. □

Notice that the property (2.3) implies that

$$|\hat{\phi}(\lambda)| \leq \|T_\phi\|, \ \forall \lambda \in \mathbb{C}, \text{ provided } \operatorname{Im}\lambda = \alpha_0.$$

Lemma 4. *Let $\phi \in C_c^\infty(\mathbb{R})$ and let λ be such that $e^{-\bar{\lambda}} \in \sigma((L_{-1})^*)$ with $Re\,\lambda = -\alpha_1$. Then we have*

$$|\hat{\phi}(i\lambda + a)| \leq \|(T_\phi)\|, \ \forall a \in \mathbb{R}. \tag{2.4}$$

Proof. Consider the semigroup $(L_{-t})^*_{t\geq 0}$ and let B be its generator. We identify H and its dual space H'. So the semigroup $(L_{-t})^*$, $t \geq 0$ is acting on H. Let $\lambda \in \mathbb{C}$ be such that $e^{-\bar{\lambda}} \in \sigma((L_{-1})^*)$ and $|e^{-\bar{\lambda}}| = \rho(L_{-1}) = \rho((L_{-1})^*) = e^{\alpha_1}$. Then, by the same argument as in Lemma 1, we prove that there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of integers and a sequence $(f_{m_k})_{k\in\mathbb{N}}$ of functions of H such that for all $t \in \mathbb{R}^+$,

$$\lim_{k\to\infty} \|(e^{tB} - e^{(-\bar{\lambda} + i2\pi n_k)t}) f_{m_k}\| = 0$$

and $\|f_{m_k}\| = 1$. Thus we deduce

$$\lim_{k\to+\infty} \|(L_{-t})^* f_{m_k} - e^{-(\bar{\lambda} - i2\pi n_k)t} f_{m_k}\| = 0, \ t \geq 0.$$

Since for $t \geq 0$ we have $L_{-t}R_t = I$, we get $(R_t)^*(L_{-t})^* = I$. Then, for $t \geq 0$ we get

$$\begin{aligned}
&\|(R_t)^* f_{m_k} - e^{(\bar{\lambda}-i2\pi n_k)t} f_{m_k}\| \\
&= \|(R_t)^* f_{m_k} - e^{(\bar{\lambda}-i2\pi n_k)t}(R_t)^*(L_{-t})^* f_{m_k}\| \\
&\leq \|(R_t)^*\| |e^{(\bar{\lambda}-i2\pi n_k)t}| \|(e^{-(\bar{\lambda}-i2\pi n_k)t} f_{m_k} - (L_{-t})^* f_{m_k})\|.
\end{aligned}$$

This implies that

$$\lim_{k\to+\infty} \|((r_+ S_t \ell_0)^* - e^{(\bar{\lambda}-i2\pi n_k)t}) f_{m_k}\| = 0, \ \forall t \in \mathbb{R}. \tag{2.5}$$

We write

$$\begin{aligned}
\hat{\phi}(i\lambda + a) &= \int_{\mathbb{R}} < \phi(t) e^{-i(a+2\pi n_k)t} f_{m_k}, e^{\bar{\lambda}t - 2i\pi n_k t} f_{m_k} > dt \\
&= \int_{\mathbb{R}} < \phi(t) e^{-i(a+2\pi n_k)t} f_{m_k}, \Big(e^{(\bar{\lambda}-i2\pi n_k)t} - (r_+ S_t \ell_0)^*\Big) f_{m_k} > dt \\
&\quad + \int_{\mathbb{R}} < \phi(t) e^{-i(a+2\pi n_k)t} (r_+ S_t \ell_0) f_{m_k}, f_{m_k} > dt = J'_k + I'_k.
\end{aligned}$$

From (2.5) we deduce that $J'_k \to 0$ as $k \to \infty$. For I'_k we apply the same argument as in the proof of Lemma 3 and we get $|\hat{\phi}(i\lambda)| \leq \|T_\phi\|$. □

Lemma 5. *For every function $\phi \in C_c^\infty(\mathbb{R})$ and for $z \in U = \{z \in \mathbb{C},\ \operatorname{Im} z \in [-\alpha_1, \alpha_0]\}$ we have*

$$|\hat{\phi}(z)| \leq \|T_\phi\|.$$

Proof. We will use the Phragmén-Lindelöf theorem and we start by proving the estimations on the bounding lines. There exists $\alpha = e^{-iz} \in \sigma(R)$ such that $|\alpha| = e^{\operatorname{Im} z} = e^{\alpha_0}$. Following (2.3), we obtain

$$|\hat{\phi}(z)| \leq \|T_\phi\|,$$

for every z such that $\operatorname{Im} z = \alpha_0$. Next notice that $\rho(L_{-1}) = \rho\Big((L_{-1})^*\Big)$. So there exists $\beta = e^{-i\bar{z}} = e^{-\overline{(-iz)}} \in \sigma((L_{-1})^*)$ such that $|\beta| = e^{\alpha_1}$ and

$$-\operatorname{Im} z = \ln|\beta| = \alpha_1.$$

Then taking into account (2.4), we get

$$|\hat{\phi}(z)| \leq \|T_\phi\|,$$

for every z such that $\operatorname{Im} z = -\alpha_1$. In the case $\alpha_1 + \alpha_0 = 0$ the result is obvious. So assume that $\alpha_0 + \alpha_1 > 0$. Since $\phi \in C_c^\infty(\mathbb{R})$ we have

$$|\hat{\phi}(z)| \leq C\|\phi\|_\infty e^{k|\operatorname{Im} z|} \leq K\|\phi\|_\infty, \ \forall z \in U,$$

where $C > 0$, $k > 0$ and $K > 0$ are constants. An application of the Phragmén-Lindelöf theorem for the holomorphic function $\widehat{\phi}$, yields

$$|\widehat{\phi}(\alpha)| \leq \|T_\phi\|$$

for $\alpha \in \{z \in \mathbb{C} : \operatorname{Im} z \in [-\alpha_1,\, \alpha_0]\}$. $\square$

Notice that for multipliers in $L^p(\mathbb{R})$ studied in [9] some relations concerning the norm $\|T_\phi\|$ and ϕ hold. Combining the results in Lemma 3-5, we get

Lemma 6. *For every $\phi \in C_c^\infty(\mathbb{R})$ and for every $a \in [-\alpha_1,\, \alpha_0]$ we have*

$$|\widehat{(\phi)_a}(x)| \leq \|T_\phi\|,\ \forall x \in \mathbb{R}.$$

Proof of Theorem 1. Let T be a Wiener-Hopf operator. Let $a \in [-\alpha_1,\, \alpha_0]$. The proof follows the approach in [12]. Notice that in [12], we establish (1.2) for $f \in C_c^\infty(\mathbb{R}^+)$ and which is new here is that we prove (1.2) for $f \in L^2_\omega(\mathbb{R}^+)$ such that $(f)_a \in L^2(\mathbb{R}^+)$. Following [12], there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ such that T is the limit of $(T_{\phi_n})_{n\in\mathbb{N}}$ with respect to the strong operator topology and we have $\|T_{\phi_n}\| \leq C\|T\|$, where C is a constant independent of n. According to Lemma 6, we have

$$|\widehat{(\phi_n)_a}(x)| \leq \|T_{\phi_n}\| \leq C\|T\|,\ \forall x \in \mathbb{R},\ \forall n \in \mathbb{N} \tag{2.6}$$

and we replace $(\widehat{(\phi_n)_a})_{n\in\mathbb{N}}$ by a suitable subsequence, also denoted by $(\widehat{(\phi_n)_a})_{n\in\mathbb{N}}$, converging with respect to the weak topology $\sigma(L^\infty(\mathbb{R}), L^1(\mathbb{R}))$ to a function $h_a \in L^\infty(\mathbb{R})$ such that $\|h_a\|_\infty \leq C\,\|T\|$. We have

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \left(\widehat{(\phi_n)_a}(x) - h_a(x)\right) g(x)\, dx = 0,\ \ \forall g \in L^1(\mathbb{R}).$$

Fix $f \in L^2_\omega(\mathbb{R}^+)$ so that $(f)_a \in L^2(\mathbb{R}^+)$. Then we get

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \left(\widehat{(\phi_n)_a}(x)\widetilde{(f)_a}(x) - h_a(x)\widetilde{(f)_a}(x)\right) g(x)\, dx = 0,$$

for all $g \in L^2(\mathbb{R})$. We conclude that $\left(\widehat{(\phi_n)_a}\widetilde{(f)_a}\right)_{n\in\mathbb{N}}$ converges weakly in $L^2(\mathbb{R})$ to $h_a\widetilde{(f)_a}$. On the other hand, we have

$$(T_{\phi_n} f)_a = \mathbf{r}_+((\phi_n)_a * \ell_0(f)_a) = \mathbf{r}_+\mathcal{F}^{-1}(\widehat{(\phi_n)_a}\widetilde{(f)_a})$$

and thus $(T_{\phi_n} f)_a$ converges weakly in $L^2(\mathbb{R}^+)$ to $\mathbf{r}_+\mathcal{F}^{-1}(h_a\widetilde{(f)_a})$. For $g \in C_c^\infty(\mathbb{R})$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^+} \left|(T_{\phi_n} f)_a(x) - (Tf)_a(x)\right| |g(x)|\, dx \\ \leq C_{a,g}\|T_{\phi_n} f - Tf\|,\ \forall n \in \mathbb{N},\end{aligned}$$

where $C_{a,g}$ is a constant depending only of g and a. Since $(T_{\phi_n} f)_{n\in\mathbb{N}}$ converges to Tf in $L^2_\omega(\mathbb{R}^+)$, we get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^+} (T_{\phi_n}\, f)_a(x) g(x)\, dx = \int_{\mathbb{R}^+} (Tf)_a(x) g(x)\, dx,\ \ \forall g \in C_c^\infty(\mathbb{R}).$$

Thus we deduce that $(Tf)_a = \mathbf{r}_+\mathcal{F}^{-1}(h_a\widetilde{(f)_a})$. The symbol h is holomorphic on $\overset{\circ}{U}$ following the same arguments as in [12]. $\square$

3. Preliminary spectral result

As a first step to our spectral analysis in this section we prove the following

Proposition 1. *Let $T \in \mathcal{M}$ and suppose that the symbol μ of T is continuous on U. Then $\mu(U) \subset \sigma(T)$.*

Proof of Proposition 1. Let T be a bounded operator on H commuting with R_t, $t \geq 0$ or L_{-t}, $t \geq 0$. For $a \in [-\alpha_1,\ \alpha_0]$, we have

$$(Tf)_a = \mathbf{r}_+\mathcal{F}^{-1}(\mu_a\widetilde{(f)_a}),\ \forall f \in L^2_\omega(\mathbb{R}^+),$$

where $\mu_a \in L^\infty(\mathbb{R})$, provided $(f)_a \in L^2(\mathbb{R}^+)$. Suppose that $\lambda \notin \sigma(T)$. Then, it follows easily that the resolvent $(T-\lambda I)^{-1}$ also commutes with $(R_t)_{t\in\mathbb{R}^+}$ or $(L_{-t})_{t\in\mathbb{R}^+}$. Consequently, $(T-\lambda I)^{-1}$ is a Wiener-Hopf operator and for $a \in [-\alpha_1,\ \alpha_0]$ there exists a function $h_a \in L^\infty(\mathbb{R})$ such that

$$((T-\lambda I)^{-1}g)_a = \mathbf{r}_+\mathcal{F}^{-1}(h_a\widetilde{(g)_a}),$$

for $g \in L^2_\omega(\mathbb{R}^+)$ such that $(g)_a \in L^2(\mathbb{R}^+)$. If f is such that $(f)_a \in L^2(\mathbb{R}^+)$, set $g = (T-\lambda I)f$. Then following Theorem 1, we deduce that $(Tf)_a \in L^2(\mathbb{R}^+)$ and $(g)_a = ((T-\lambda I)f)_a \in L^2(\mathbb{R}^+)$. Thus applying once more Theorem 1, we get

$$\begin{aligned}((T-\lambda I)^{-1}(T-\lambda I)f)_a &= \mathbf{r}_+\mathcal{F}^{-1}(h_a\widetilde{(T-\lambda I)f})_a\\ &= \mathbf{r}_+\mathcal{F}^{-1}\Big(h_a\mathcal{F}\mathbf{r}_+[\mathcal{F}^{-1}((\mu_a-\lambda)\widetilde{(f)_a})]\Big).\end{aligned}$$

We have

$$\|(f)_a\|_{L^2} \leq \|h_a\mathcal{F}\mathbf{r}_+\mathcal{F}^{-1}((\mu_a-\lambda)\widetilde{(f)_a})\|_{L^2} \leq \|h_a\|_\infty\|\mathcal{F}\mathbf{r}_+\mathcal{F}^{-1}((\mu_a-\lambda)\widetilde{(f)_a})\|_{L^2}$$

and we deduce

$$\|\widetilde{(f)_a}\|_{L^2} \leq C\|(\mu_a-\lambda)\widetilde{(f)_a}\|_{L^2}, \tag{3.1}$$

for all $f \in L^2_\omega(\mathbb{R}^+)$ such that $(f)_a \in L^2(\mathbb{R}^+)$. Let $\lambda = \mu_a(\eta_0) = \mu(\eta_0+ia) \in \mu(U)$ for $a \in [-\ln\rho(L_{-1}),\ \ln\rho(R)]$ and some $\eta_0 \in \mathbb{R}$. Since the symbol μ of T is continuous, the function $\mu_a(\eta) = \mu(\eta+ia)$ is continuous on $\mathbb{R}$. We will construct a function $f(x) = F(x)e^{-ax}$ with $\mathrm{supp}(F) \subset \mathbb{R}^+$ for which (3.1) is not fulfilled. Consider

$$g(t) = e^{-\frac{b^2(t-t_0)^2}{2}}e^{i(t-t_0)\eta_0},\ b>0, t_0>1$$

with Fourier transform

$$\hat{g}(\xi) = \frac{1}{b}e^{-\frac{(\xi-\eta_0)^2}{2b^2}}e^{-it_0\xi}.$$

Fix a small $0<\epsilon<\frac{1}{2}C^{-2}$, where C is the constant in (3.1) and let $\delta>0$ be fixed so that $|\mu_a(\xi)-\lambda| \leq \sqrt{\epsilon}$ for $\xi \in V = \{\xi \in \mathbb{R} :\ |\xi-\eta_0| \leq \delta\}$. Moreover, assume that

$$|\mu_a(\xi)-\lambda|^2 \leq C_1,\ a.e.\ \xi \in \mathbb{R}.$$

We have for $0<b\leq 1$ small enough

$$\int_{\mathbb{R}\setminus V}|\hat{g}(\xi)|^2d\xi \leq \frac{1}{b^2}\int_{|\xi-\eta_0|\geq\delta}e^{-\frac{(\xi-\eta_0)^2}{2b^2}}d\xi$$

$$\leq e^{-\frac{\delta^2}{4b^2}}\frac{1}{b^2}\int_{|\xi-\eta_0|\geq\delta} e^{-\frac{(\xi-\eta_0)^2}{4b^2}}d\xi \leq C_0 b^{-1}e^{-\frac{\delta^2}{4b^2}} \leq \epsilon$$

with $C_0>0$ independent of $b>0$. We fix $b>0$ with the above property and we choose a function $\varphi\in C_c^\infty(\mathbb{R}^+)$ such that $0\leq\varphi\leq 1$, $\varphi(t)=1$ for $1\leq t\leq 2t_0-1$, $\varphi(t)=0$ for $t\leq 1/2$ and for $t\geq 2t_0-1/2$. We suppose that $|\varphi^{(k)}(t)|\leq c_1$, $k=1,2,\forall t\in\mathbb{R}$. Set $G(t)=(\varphi(t)-1)g(t)$. We will show that

$$|(1+\xi^2)\hat{G}(\xi)|\leq\sqrt{\frac{C_2}{4\pi}\epsilon} \tag{3.2}$$

for t_0 large enough with $C_2>0$ independent of t_0. On the support of $(\varphi-1)$ we have $|t-t_0|>t_0-1$ and integrating by parts in $\int_{\mathbb{R}}(1+\xi^2)G(t)e^{-it\xi}dt$ we must estimate the integral

$$\int_{|t-t_0|\geq t_0-1} e^{-\frac{b^2(t-t_0)^2}{2}}(1+|t-t_0|+(t-t_0)^2)dt$$
$$\leq\Big(\int_{-\infty}^{1-t_0}(1+|y|+y^2)e^{-b^2y^2/2}dy+\int_{t_0-1}^{\infty}(1+y+y^2)e^{-b^2y^2/2}dy\Big).$$

Choosing t_0 large enough we arrange (3.2).

We set $F=\varphi g\in C_c^\infty(\mathbb{R}^+)$ and we obtain

$$\int_{\mathbb{R}\setminus V}|\tilde{F}(\xi)|^2d\xi\leq 2\int_{\mathbb{R}\setminus V}|\hat{g}(\xi)|^2d\xi+2\int_{\mathbb{R}\setminus V}|\hat{G}(\xi)|^2d\xi$$
$$\leq 2\epsilon+\frac{C_2\epsilon}{2\pi}\int_{\mathbb{R}}(1+\xi^2)^{-2}d\xi\leq(2+C_2)\epsilon.$$

Then

$$\int_{\mathbb{R}}|(\mu_a(\xi)-\lambda)\tilde{F}(\xi)|^2d\xi\leq\int_{\mathbb{R}\setminus V}|(\mu_a(\xi)-\lambda)\tilde{F}(\xi)|^2d\xi+\int_V|(\mu_a(\xi)-\lambda)\tilde{F}(\xi)|^2d\xi$$
$$\leq C_1(2+C_2)\epsilon+(2\pi)^2\|F\|_{L^2}^2\epsilon.$$

Now assume (3.1) fulfilled. Therefore

$$(2\pi)^2\|F\|_{L^2}^2\leq C^2\|(\mu_a(\xi)-\lambda)\hat{F}(\xi)\|_{L^2}^2\leq C^2C_1(2+C_2)\epsilon+(2\pi C)^2\|F\|_{L^2}^2\epsilon,$$

and since $C^2\epsilon<\frac{1}{2}$, we conclude that

$$\|F\|_{L^2}^2\leq\frac{C^2C_1}{2\pi^2}(2+C_2)\epsilon.$$

On the other hand,

$$\|F\|_{L^2}^2\geq\frac{1}{2}\|g\|_{L^2}^2-\|(\varphi-1)g\|_{L^2}^2\geq\frac{1}{2}\|g\|_{L^2}^2-(2\pi)^{-2}\frac{C_2}{2}\epsilon$$

and

$$\int_{\mathbb{R}}|g(t)|^2dt\geq\int_{|t-t_0|\leq\frac{1}{b}}e^{-b^2(t-t_0)^2}dt\geq\frac{2e^{-1}}{b}\geq 2e^{-1}.$$

For small ϵ we obtain a contradiction, since C_2 is independent of ϵ. This completes the proof. □

4. Spectra of $(R_t)_{t\in\mathbb{R}^+}$, $(L_{-t})_{t\in\mathbb{R}^+}$ and bounded operators commuting with at least one of these semigroups

Observing that the symbol of R_t is $z \longrightarrow e^{-itz}$, an application of Proposition 1 to the operator R_t yields

$$\{z \in \mathbb{C},\ e^{-\alpha_1 t} \leq |z| \leq e^{\alpha_0 t}\} \subset \sigma(R_t). \tag{4.1}$$

This inclusion describes only a part of the spectrum of R_t. We will show that in our general setting we have (1.3). To prove this, for $t > 0$ assume that $z \in \mathbb{C}$ is such that $0 < |z| < e^{-\alpha_1 t}$. Let $g \in H$ be a function such that $g(x) = 0$ for $x \geq t$ and $g \neq 0$. If the operator $(zI - R_t)$ is surjective on H, then there exists $f \neq 0$ such that $(z - R_t)f = g$. This implies $L_{-t}g = 0$ and hence

$$\Big(L_{-t} - \frac{1}{z}I\Big)f = 0$$

which is a contradiction. So every such z is in the spectrum of R_t and we obtain (1.3).

Next, it is easy to see that in our setup for the approximative point spectrum $\Pi(R_t)$ of R_t we have the inclusion

$$\Pi(R_t) \subset \{z \in \mathbb{C} :\ e^{-\alpha_1 t} \leq |z| \leq e^{\alpha_0 t}\}. \tag{4.2}$$

Indeed, for $z \neq 0$, we have the equality

$$R_{-t} - \frac{1}{z}I = \frac{1}{z}R_{-t}(zI - R_t).$$

If for $z \in \mathbb{C}$ with $0 < |z| < e^{-\alpha_1 t}$, there exists a sequence (f_n) such that $\|f_n\| = 1$ and $\|(zI - R_t)f_n\| \to 0$ as $n \to \infty$, then

$$\Big(L_{-t} - \frac{1}{z}I\Big)f_n \to 0,\ n \to \infty$$

and this leads to $\frac{1}{z} \in \sigma(L_{-t})$ which is a contradiction. Next, if $0 \in \Pi(R_t)$, there exists a sequence $g_n \in H$ such that $R_t g_n \to 0$, $\|g_n\| = 1$. Then $g_n = L_{-t}R_t g_n$ and we obtain a contradiction. Since the symbol of L_{-t} is $z \longrightarrow e^{itz}$, applying Proposition 1, we obtain

$$\{z \in \mathbb{C} :\ e^{-\alpha_0 t} \leq |z| \leq e^{\alpha_1 t}\} \subset \sigma(L_{-t}).$$

Passing to the proof of (1.4), notice that $R_t^*(L_{-t})^* = I$. Then for $0 < |z| < e^{-\alpha_0 t}$ we have

$$z\Big(\frac{1}{z}I - R_t^*\Big) = R_t^*\Big((L_{-t})^* - zI\Big). \tag{4.3}$$

It is clear that $0 \in \sigma_r(R_t)$, where $\sigma_r(R_t)$ denotes the residual spectrum of R_t. In fact, if $0 \notin \sigma_r(R_t)$, then 0 is in the approximative point spectrum of R_t and this contradicts (4.2). Since $0 \in \sigma_r(R_t)$, we deduce that 0 is an eigenvalue of R_t^*. Let $R_t^* g = 0$, $g \neq 0$. Assume that $(L_{-t})^* - zI$ is surjective. Therefore, there exists $f \neq 0$ so that $((L_{-t})^* - z)f = g$ and (4.3) yields $(\frac{1}{z} - R_t^*)f = 0$. Consequently, $\frac{1}{|z|} \leq \rho(R_t^*) = \rho(R_t) = e^{\alpha_0 t}$ and we obtain a contradiction. Thus we conclude that $z \in \sigma((L_{-t})^*)$, hence $\bar{z} \in \sigma(L_{-t})$ and the proof

of (1.4) is complete.

To study the operators commuting with $(R_t)_{t\in\mathbb{R}^+}$, we need the following

Lemma 7. *Let $\phi \in C_c^\infty(\mathbb{R})$. The operator T_ϕ commutes with R_t, $\forall t > 0$, if and only if the support of ϕ is in $\overline{\mathbb{R}^+}$.*

Proof. First if $\psi \in L^2_\omega(\mathbb{R}^+)$ has compact support in $\overline{\mathbb{R}^+}$, it is easy to see that T_ψ commutes with R_t, $t \geq 0$. Now consider $\phi \in C_c^\infty(\mathbb{R})$ and suppose that T_ϕ commutes with R_t, $t \geq 0$. We write $\phi = \phi\chi_{\mathbb{R}^-} + \phi\chi_{\mathbb{R}^+}$. If T_ϕ commutes with R_t, $t \geq 0$, then the operator $T_{\phi\chi_{\mathbb{R}^-}}$ commutes too. Let the function $\psi = \phi\chi_{\mathbb{R}^-}$ have support in $[-a, 0]$ with $a > 0$. Setting $f = \chi_{[0,a]}$, we get $S_a f = \chi_{[a,2a]}$. For $x \geq 0$ we have

$$T_\psi R_a r_+ f(x) = \mathbf{r}_+(\psi * S_a f)(x) = \int_{-a}^{0} \psi(t)\chi_{\{a\leq x-t\leq 2a\}}dt = \int_{max(-a,-2a+x)}^{min(x-a,0)} \psi(t)dt.$$

Since $T_\psi R_a = R_a T_\psi$, we deduce $\int_{-a}^{x-a} \psi(t)dt = 0$, $\forall x \in [0, a]$. This implies that $\psi(t) = 0$, for $t \in [-a, 0]$ and $supp(\phi) \subset \overline{\mathbb{R}^+}$. □

Lemma 8. *Let λ be such that $e^\lambda \in \sigma(R)$. Then there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of integers and a sequence $(f_{m_k})_{k\in\mathbb{N}}$ of functions of H such that*

$$\lim_{k\to\infty} < \Big(R_t - e^{(\lambda+2\pi i n_k)t}\Big) f_{m_k}, f_{m_k} >= 0,\ \forall t \in \mathbb{R}^+,\ \ \|f_{m_k}\| = 1,\ \forall k \in \mathbb{N}. \tag{4.4}$$

Proof. If $\lambda \notin \sigma(A)$, we repeat the argument of the proof of Lemma 1. Denote by $\sigma_r(A)$ the residual spectrum of A. If $\lambda \in \sigma(A) \setminus \sigma_r(A)$, we deduce that λ is in the approximative point spectrum of A and we can apply the argument of Lemma 1. Finally, if $\lambda \in \sigma_r(A)$, then there exists $f \in H$ such that $A^* f = \overline{\lambda} f$ and $\|f\| = 1$. We set $f_{m_k} = f$, $n_k = 0$, for $k \in \mathbb{N}$ and we use the fact that $(R_t)^* f = e^{\bar{\lambda}t} f$. □

Lemma 9. *For all $\phi \in C_c^\infty(\overline{\mathbb{R}^+})$ and λ such that $e^\lambda \in \sigma(R)$ we have*

$$|\hat{\phi}(i\lambda)| \leq \|T_\phi\|. \tag{4.5}$$

The proof is based on the equality

$$\hat{\phi}(i\lambda) = \int_{\mathbb{R}^+} \phi(t)e^{\lambda t}dt = \int_{\mathbb{R}^+} \langle \phi(t)e^{(\lambda+2\pi i n_k)t} f_{m_k}, e^{2\pi i n_k t} f_{m_k}\rangle dt$$

$$= \int_{\mathbb{R}^+} \langle \phi(t)\Big(e^{(\lambda+2\pi i n_k)t} I - R_t\Big) f_{m_k}, e^{2\pi i n_k t} f_{m_k}\rangle dt + \int_{\mathbb{R}^+} \langle \phi(t) R_t f_{m_k}, e^{2\pi i n_k t} f_{m_k}\rangle dt.$$

We apply Lemma 8 and we repeat the argument of the proof of Lemma 3. Notice that here the integration is over $\mathbb{R}^+$ and we do not need to examine the integral for $t < 0$. Following [12], the operator T is a limit of a sequence of operators T_{ϕ_n}, where $\phi_n \in C_c^\infty(\mathbb{R})$ and $\|T_{\phi_n}\| \leq C\|T\|$. The sequence $(T_{\phi_n})_{n\geq 0}$ has been constructed in [12] and it follows from its construction that if T commutes with R_t, $t > 0$, then T_{ϕ_n} has the same property for all $n \in \mathbb{N}$. Therefore, Lemma 7 implies that $\phi_n \in C_c^\infty(\overline{\mathbb{R}^+})$ and to obtain Theorem 2 for bounded operators commuting with $(R_t)_{t>0}$, we apply Lemma 9 and the same

arguments as in the proof of Theorem 1. Finally, applying Theorem 2 and the arguments of the proof of Proposition 1, we establish (1.5) and this completes the proof of iii) in Theorem 3. Next we prove the following

Lemma 10. *Let $\phi \in C_c^\infty(\mathbb{R})$. Then T_ϕ commutes with L_{-t}, $\forall t > 0$ if and only if $supp(\phi) \subset \overline{\mathbb{R}^-}$.*

The proof of Lemma 10 is essentially the same as that of Lemma 7. By using Lemma 10, we obtain an analogue of Lemma 9 and Theorem 2 for bounded operators commuting with $(L_{-t})_{t>0}$ and applying these results we establish iv) in Theorem 3.

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LMAM, Université de Lorraine (Metz), UMR 7122,Ile du Saulcy 57045, Metz Cedex 1, France.
E-mail address: petkova@univ-metz.fr