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Wiener-Hopf operators on $L^2_w(\mathbb{R}^+)$

By

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Abstract. Let $L^2_\omega(\mathbb{R}^+)$ be a weighted space with weight ω. In this paper we show that for every Wiener-Hopf operator T on $L^2_\omega(\mathbb{R}^+)$ and for every $a \in I_\omega$, there exists a function $\nu_a \in L^\infty(\mathbb{R})$ such that

$$(Tf)_a = P^+\mathcal{F}^{-1}(\nu_a \widehat{(f)_a}),$$

for all $f \in C_c^\infty(\mathbb{R}^+)$. Here $(g)_a$ denotes the function $x \longrightarrow g(x)e^{ax}$ for $g \in L^2_\omega(\mathbb{R}^+)$, $P^+ f = \chi_{\mathbb{R}^+} f$ and $I_\omega = [\ln R^-_\omega, \ln R^+_\omega]$, where R^+_ω is the spectral radius of the shift $S : f(x) \longrightarrow f(x-1)$ on $L^2_\omega(\mathbb{R}^+)$, while $\frac{1}{R^-_\omega}$ is the spectral radius of the backward shift $S^{-1} : f(x) \longrightarrow (P^+ f)(x+1)$ on $L^2_\omega(\mathbb{R}^+)$. Moreover, there exists a constant C_ω, depending on ω, such that $\|\nu_a\|_\infty \leqq C_\omega \|T\|$ for every $a \in I_\omega$. If $R^-_\omega < R^+_\omega$, we prove that there exists a bounded holomorphic function ν on $\overset{\circ}{A}_\omega := \{z \in \mathbb{C} \mid \operatorname{Im} z \in \overset{\circ}{I}_\omega\}$ such that for $a \in \overset{\circ}{I}_\omega$, the function ν_a is the restriction of ν on the line $\{z \in \mathbb{C} \mid \operatorname{Im} z = a\}$.

1. Introduction. Let ω be a weight on $\mathbb{R}^+ := [0, +\infty[$, i.e. a positive measurable function on $\mathbb{R}^+$ satisfying

$$0 < \operatorname{ess} \inf_{x \geqq 0} \frac{\omega(x+y)}{\omega(x)} \leqq \operatorname{ess} \sup_{x \geqq 0} \frac{\omega(x+y)}{\omega(x)} < +\infty, \ \forall y \in \mathbb{R}^+. \tag{1.1}$$

The purpose of this paper is to study the representation of Wiener-Hopf operators on the space $L^2_\omega(\mathbb{R}^+) := \{f \text{ measurable on } \mathbb{R}^+ \mid \int_0^{+\infty} |f(x)|^2 \omega(x)^2 dx < +\infty\}$. We will consider $L^2_\omega(\mathbb{R}^+)$ as a subspace of $L^2(\mathbb{R}^-) \oplus L^2_\omega(\mathbb{R}^+)$ by setting $f(t) = 0$, for $t < 0$, when $f \in L^2_\omega(\mathbb{R}^+)$. The space $L^2_\omega(\mathbb{R}^+)$ is a Hilbert space with respect to the sesquilinear form

$$\langle f, g \rangle := \langle f, g \rangle_\omega = \int_{\mathbb{R}^+} f(x)\overline{g}(x)\omega(x)^2 dx, \ \forall f \in L^2_\omega(\mathbb{R}^+), \ \forall g \in L^2_\omega(\mathbb{R}^+).$$

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We will denote by $S_{a,\omega}$ the translation operator from $L^2(\mathbb{R}^-) \oplus L^2_\omega(\mathbb{R}^+)$ to $L^2(\mathbb{R}^-) \oplus L^2_\omega(\mathbb{R}^+)$ defined by

$$(S_{a,\omega} f)(x) = f(x-a),$$

for $a \in \mathbb{R}$, $x \in \mathbb{R}$. Set

$$\tilde{\omega}(x) = \operatorname{ess}\sup_{y \geqq 0} \frac{\omega(x+y)}{\omega(y)}, \text{ for } x \geqq 0,$$

$$\tilde{\omega}(x) = \operatorname{ess}\sup_{y \geqq 0} \frac{\omega(y)}{\omega(y-x)}, \text{ for } x < 0,$$

and denote by P^+ the operator from $L^2(\mathbb{R}^-) \oplus L^2_\omega(\mathbb{R}^+)$ to $L^2_\omega(\mathbb{R}^+)$ defined by $P^+ f = \chi_{\mathbb{R}^+} f$. We have

$$\|S_{a,\omega} P^+\| = \tilde{\omega}(a), \forall a \geqq 0$$

and

$$\|P^+ S_{a,\omega} P^+\| = \tilde{\omega}(a), \forall a < 0.$$

When there is no risk of confusion, we will write S_a instead of $S_{a,\omega}$. Denote by $B(X)$ the set of bounded operators on the space X.

Definition 1. An operator $T \in B(L^2_\omega(\mathbb{R}^+))$ is called a Wiener-Hopf operator if

$$P^+ S_{-a} T S_a f = Tf, \text{ for all } a \in \mathbb{R}^+,\ f \in L^2_\omega(\mathbb{R}^+).$$

Denote by W_ω the space of Wiener-Hopf operators on $L^2_\omega(\mathbb{R}^+)$ and denote by $C_c^\infty(\mathbb{R}^+)$ the space of functions in $C^\infty(\mathbb{R})$ with compact support in $\mathbb{R}^+$. The case $\omega = 1$ is well known (see [3]). Indeed, for every $T \in W_1$, there exists a distribution μ_T such that

$$Tf = P^+(\mu_T * f), \text{ for } f \in C_c^\infty(\mathbb{R}^+). \tag{1.2}$$

Moreover, there exists a function $h \in L^\infty(\mathbb{R})$, called the symbol of T, such that

$$Tf = P^+ \mathcal{F}^{-1}(h\hat{f}), \text{ for } f \in L^2(\mathbb{R}^+). \tag{1.3}$$

This paper is devoted to a generalisation of the results (1.2) and (1.3) for $T \in W_\omega$, where ω is a function satisfying only (1.1). We are motivated by a recent result of Jean Esterle, who proved in [2] that a Toeplitz operator on $l^2_\sigma(\mathbb{Z}^+) := \{(u_n)_{n \geqq 0} \mid \sum_{n \geqq 0} |u_n|^2 \sigma(n)^2 < +\infty\}$ is associated to a bounded function on the set $U := \{z \in \mathbb{C} \mid \frac{1}{\rho(T)} \leqq |z| \leqq \rho(S)\}$, where S and T denote respectively the shift and the backward shift on $l^2_\sigma(\mathbb{Z}^+)$ and $\rho(A)$ denotes the spectral radius of the operator A. Moreover, this function is holomorphic on $\mathring{U}$, if $\mathring{U} \neq \emptyset$. On the other hand, in a recent paper (see [5]), the author showed that every multiplier (bounded operator commuting with translations) on a weighted space $L^2_\delta(\mathbb{R}) := \{f \text{ measurable on } \mathbb{R} \mid \int_{-\infty}^{+\infty} |f(x)|^2\delta(x)^2dx < +\infty\}$ has the representation $\widehat{Tf} = h\hat{f}$, for $f \in C_c^\infty(\mathbb{R})$, on a band $\Omega_\delta \subset \mathbb{C}$ determined by δ. Here h is a L^∞ function on the boundary of Ω_δ, h is bounded and holomorphic on $\overset{\circ}{\Omega}_\delta$, if $\overset{\circ}{\Omega}_\delta \neq \emptyset$, and the weight δ satisfies a condition similar to (1.1). To our best knowledge there are no general results concerning the representation of Wiener-Hopf operators on $L^2_\omega(\mathbb{R}^+)$. Taking into account the similarities between Wiener-Hopf operators and multipliers and the results of [5] and [2], it is natural to conjecture that Wiener-Hopf operators have representation analogous to (1.3). Nevertheless, there are some important differences and it is not yet known if every Wiener-Hopf operator on a general weighted space $L^2_\omega(\mathbb{R}^+)$ can be extended as a multiplier on some weighted space $L^2_\delta(\mathbb{R})$. Every Wiener-Hopf operator on $L^2(\mathbb{R}^+)$ is given by P^+M, where M is a multiplier on $L^2(\mathbb{R})$ (see [3]) and then (1.2) and (1.3) follow obviously from the results in [4]. In the general case, the argument of [3] is inapplicable and it seems difficult to show that every Wiener-Hopf operator is induced by a multiplier. Despite of this open question, inspired by methods developed in [5], we obtain the result below. Set

$$R_\omega^+ = \lim_{n\to+\infty} \tilde{\omega}(n)^{\frac{1}{n}},\ R_\omega^- = \lim_{n\to+\infty} \tilde{\omega}(-n)^{-\frac{1}{n}},$$

$$I_\omega := [\ln R_\omega^-, \ln R_\omega^+],\ A_\omega := \{z \in \mathbb{C} \mid \operatorname{Im} z \in I_\omega\},$$

$$C_\omega = \exp \int_1^2 2 \ln \tilde{\omega}(u)du.$$

Theorem 1. *Let ω be a weight on $\mathbb{R}^+$ and let $T \in W_\omega$. Then*

1) *For all $a \in I_\omega$ we have $(Tf)_a \in L^2(\mathbb{R}^+)$, for $f \in C_c^\infty(\mathbb{R}^+)$.*
2) *For all $a \in I_\omega$ there exists a function $\nu_a \in L^\infty(\mathbb{R})$ such that*

$$(Tf)_a = P^+\mathcal{F}^{-1}(\nu_a\widehat{(f)_a}),\ for\ f \in C_c^\infty(\mathbb{R}^+).$$

3) *Moreover, if $\overset{\circ}{I}_\omega \neq \emptyset$ ($R_\omega^- < R_\omega^+$), there exists a function $\nu \in \mathcal{H}^\infty(\overset{\circ}{A}_\omega)$ such that for all $a \in \overset{\circ}{I}_\omega$*

$$\nu(x + ia) = \nu_a(x),\ almost\ everywhere\ on\ \mathbb{R}^+$$

and we have $\|\nu\|_\infty \leqq C_\omega\|T\|$.

Notice that following the argument of [1], we can show as in [5], that the weight ω is equivalent to a continuous weight ω_0 defined by

$$\omega_0(x) = \exp\left(\int_1^2 \ln(\omega(x+t))dt\right).$$

Moreover, ω_0 is such that $\ln \omega_0$ is a Liptchitz function. This implies

$$\lim_{n\to+\infty} \sup_{0 \leqq t \leqq \frac{1}{n}} \tilde{\omega_0}(t) = 1$$

and for every compact set $K \subset \mathbb{R}$, we have

$$\sup_{t\in K} \tilde{\omega}(t) < +\infty.$$

Hence, if $K \subset \mathbb{R}^+$, then

$$0 < \inf_{x\in K} \omega(x) \leqq \sup_{x\in K} \omega(x) < +\infty.$$

It is clear that $A_\omega = A_{\omega_0}$. In the same way, as in [5], we observe that if $T \in B(L^2_\omega(\mathbb{R}^+))$ we have

$$\|T\| = \sup_{\substack{f\in L^2_\omega(\mathbb{R}^+)\\ f\neq 0}} \frac{\|Tf\|_{\omega_0}}{\|f\|_{\omega_0}} \leqq C_\omega \sup_{\substack{f\in L^2_\omega(\mathbb{R}^+)\\ f\neq 0}} \frac{\|Tf\|_\omega}{\|f\|_\omega}.$$

Thus it is sufficient to prove Theorem 1 for a weight having the properties of ω_0 and we obtain the result for ω with a modification of the estimation of the norm of the symbol. First, we generalise (1.2) in Section 2, by using an appropriate definition of μ_T and the methods of [4]. In Section 3 we approximate a Wiener-Hopf operator expointing the arguments of [5]. In Section 4, we prove Theorem 1.

2. Distribution associated to a Wiener-Hopf operator. In this section we prove that every Wiener-Hopf operator is associated to a distribution. Denote by $C_0^\infty(\mathbb{R}^+)$ the space of functions of $C^\infty(\mathbb{R})$ with support in $]0, +\infty[$. Set $H^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \mid f' \in L^2(\mathbb{R})\}$, the derivative of $f \in L^2(\mathbb{R})$ being taken in the sense of distributions.

Lemma 1. *If $T \in W_\omega$ and $f \in C_0^\infty(\mathbb{R}^+)$, then $(Tf)' = T(f')$.*

Proof. Let $f \in C_0^\infty(\mathbb{R}^+)$ and let $(h_n)_{n\geqq 0} \subset \mathbb{R}^+$ be a sequence converging to 0. We have

$$\left| \frac{(S_{-h_n} f)(x) - f(x)}{h_n} - f'(x) \right| \leqq 2\|f'\|_\infty, \forall x \in \mathbb{R}^+$$

and by using the dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} \left\| \frac{P^+ S_{-h_n} f - f}{h_n} - f' \right\|_\omega = 0.$$

Next we get

$$\lim_{n\to+\infty} \left\| \frac{TP^+ S_{-h_n} f - Tf}{h_n} - T(f') \right\|_\omega = 0.$$

Since $T \in W_\omega$, this implies for $n \gg 1$

$$TP^+S_{-h_n}f = TS_{-h_n}f = P^+S_{-h_n}TS_{h_n}S_{-h_n}f = P^+S_{-h_n}Tf.$$

Then we have

$$\lim_{n\to+\infty}\int_0^{+\infty}\left|\frac{(Tf)(x+h_n)-(Tf)(x)}{h_n} - T(f')(x)\right|^2 \omega(x)^2dx = 0.$$

It follows that $\frac{P^+S_{-h_n}Tf-Tf}{h_n}$ converges to $T(f')$ in the sense of distributions and $T(f') = (Tf)'$. □

Denote by $C_K^\infty(\mathbb{R})$ the space of functions of $C_c^\infty(\mathbb{R})$ with support in the compact K.

Theorem 2. *If T is a Wiener-Hopf operator, then there exists a distribution μ_T such that*

$$Tf = P^+(\mu_T * f),$$

for $f \in C_c^\infty(\mathbb{R}^+)$.

Proof. Set $\tilde{f}(x) = f(-x)$, for $f \in C_c^\infty(\mathbb{R})$, $x \in \mathbb{R}$. Let $f \in C_c^\infty(\mathbb{R})$ and let z_f be such that supp $\tilde{f} \subset]-z_f, +\infty[$ and $S_z\tilde{f} \in C_0^\infty(\mathbb{R}^+)$ for $z \geqq z_f$. We have $(TS_z\tilde{f})' = T(S_z\tilde{f})'$ and $(TS_z\tilde{f})' \in L^2_{\text{loc}}(\mathbb{R})$. It follows that $TS_z\tilde{f}$ coincides with a continuous function on $\mathbb{R}^+$ (see [6, p. 186]). Moreover, for $a > 0$ and $z \geqq z_f$ we have

$$(TS_{z+a}\tilde{f})(z+a) = (P^+S_{-a}TS_a(S_z\tilde{f}))(z) = (TS_z\tilde{f})(z).$$

Thus we conclude that $\{(TS_z\tilde{f})(z)\}_{z\in\mathbb{R}^+}$ is a constant for $z \geqq z_f$ and we set

$$\langle \mu_T, f\rangle = \lim_{z\to+\infty}(TS_z\tilde{f})(z).$$

Let K be a compact subset of $\mathbb{R}$ and let z_K be such that $z_K \geqq 1$ and $K \subset]-\infty, z_K[$. Choose $g \in C_c^\infty(\mathbb{R})$ such that g is positive, supp $g \subset [z_K - 1, z_K + 1]$ and $g(z_K) = 1$. For $f \in C_K^\infty(\mathbb{R})$, we have $gT(S_{z_K}\tilde{f}) \in H^1(\mathbb{R})$ and it follows from Sobolev's lemma (see [6]) that

$$\begin{aligned}|(TS_{z_K}\tilde{f})(z_K)| &= |g(z_K)(TS_{z_K}\tilde{f})(z_K)| \\ &\leqq C\left(\left(\int_{|y-z_K|\leqq 1} g(y)^2|(TS_{z_K}\tilde{f})(y)|^2dy\right)^{\frac{1}{2}} \right.\\ &\quad \left. + \left(\int_{|y-z_K|\leqq 1} |(g(TS_{z_K}\tilde{f}))'(y)|^2dy\right)^{\frac{1}{2}}\right),\end{aligned}$$

where $C > 0$ is a constant. It implies that there exists a constant $C(K)$, depending only on K, such that

$$|(TS_{z_K}\tilde{f})(z_K)| \leqq C(K)\left(\left(\int\limits_{|y-z_K|\leqq 1}|(TS_{z_K}\tilde{f})(y)|^2\frac{\omega(y)^2}{\omega(y)^2}dy\right)^{\frac{1}{2}} + \left(\int\limits_{|y-z_K|\leqq 1}|(T(S_{z_K}\tilde{f})')(y)|^2\frac{\omega(y)^2}{\omega(y)^2}dy\right)^{\frac{1}{2}}\right).$$

Since $\sup\limits_{t\in[z_K-1,z_K+1]}\frac{\mathbf{1}}{\omega(t)} < +\infty$ and $\sup\limits_{t\in[z_K-1,z_K+1]}\omega(t) < +\infty$, it follows that for $f \in C_K^\infty(\mathbb{R})$ we have

$$\begin{aligned}
&|(TS_{z_K}\tilde{f})(z_K)| \\
&\leqq \mathcal{C}(K)\|T\|\left(\left(\int\limits_{|y-z_K|\leqq M}|(S_{z_K}\tilde{f})(y)|^2dy\right)^{\frac{1}{2}} + \left(\int\limits_{|y-z_K|\leqq M}|(S_{z_K}\tilde{f})'(y)|^2dy\right)^{\frac{1}{2}}\right) \\
&\leqq \mathcal{C}(K)\|T\|\left(\left(\int_{-M}^{M}|\tilde{f}(x)|^2dx\right)^{\frac{1}{2}} + \left(\int_{-M}^{M}|(\tilde{f})'(x)|^2dx\right)^{\frac{1}{2}}\right) \\
&\leqq \mathcal{C}(K)\|T\|(\|\tilde{f}\|_\infty + \|\tilde{f}'\|_\infty) = \mathcal{C}(K)\|T\|(\|f\|_\infty + \|f'\|_\infty),
\end{aligned}$$

where $\mathcal{C}(K)$ is a constant depending only on K. Since for all $z \geqq z_K$ and for $f \in C_K^\infty(\mathbb{R})$ we have

$$(TS_z\tilde{f})(z) = (TS_{z_K}\tilde{f})(z_K),$$

we deduce that μ_T is a distribution. On the other hand, for $y \geqq 0$ and $f \in C_c^\infty(\mathbb{R}^+)$ we have for $z > y$:

$$\begin{aligned}
(Tf)(y) &= (S_{-y}Tf)(0) = (S_{-y}S_{-z}TS_zf)(0) \\
&= (S_{-z}(S_{-y}TS_y)S_{-y}S_zf)(0) = (S_{-z}TS_{-y}S_zf)(0) \\
&= (TS_zS_{-y}f)(z).
\end{aligned}$$

Consequently,

$$\lim_{z\to+\infty}(TS_zS_{-y}f)(z) = (Tf)(y).$$

Next, we have, for $y \geqq 0$ and for $f \in C_c^\infty(\mathbb{R}^+)$,

$$\lim_{z\to+\infty}(TS_zS_{-y}f)(z) = \langle \mu_T, \widetilde{S_{-y}f}\rangle = \langle \mu_{T,x}, f(y-x)\rangle$$
$$= (\mu_T * f)(y)$$

and we conclude that

$$(Tf)(y) = (\mu_T * f)(y),\ y \geqq 0,\ f \in C_c^\infty(\mathbb{R}^+).$$

□

3. Approximation of Wiener-Hopf operators. In this section we will apply the arguments of Section 3 in [5] with some modifications. For the convenience of the reader we detail the proofs.

Denote by T_μ the Wiener-Hopf operator defined by the convolution with μ for $f \in C_c^\infty(\mathbb{R}^+)$. If μ has compact support, then T_μ will be called a Wiener-Hopf operator with compact support.

Theorem 3. *Let ω be a weight on $\mathbb{R}^+$ and let $T \in W_\omega$. Then there exists a sequence $(Y_n)_{n\in\mathbb{N}}$ of Wiener-Hopf operators with compact support such that*

$$\lim_{n\to+\infty}\|Y_nf - Tf\|_\omega = 0,\ for\ f \in L^2_\omega(\mathbb{R}^+)$$

and

$$\|Y_n\| \leqq \|T\|, \forall n \in \mathbb{N}.$$

Proof. Set $(M_tf)(x) = f(x)e^{-itx}$, for $f \in L^2_\omega(\mathbb{R}^+)$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^+$. By using the dominated convergence theorem, we obtain that the group $(M_t)_{t\in\mathbb{R}}$ is continuous with respect to the strong operator topology. Let $T \in W_\omega$ and set $\mathcal{T}(t) = M_{-t}\circ T\circ M_t$, $\forall t \in \mathbb{R}$. For $a > 0$, $x > 0$ and $f \in L^2_\omega(\mathbb{R}^+)$ we have

$$\begin{aligned}(S_{-a}\mathcal{T}(t)S_af)(x) &= (\mathcal{T}(t)S_af)(x+a)\\ &= e^{it(x+a)}(T(f(s-a)e^{-its}))(x+a)\\ &= e^{itx}(S_{-a}T(f(s-a)e^{-it(s-a)}))(x)\\ &= e^{itx}(S_{-a}TS_a(M_tf))(x) = (\mathcal{T}(t)f)(x).\end{aligned}$$

This shows that $\mathcal{T}(t) \in W_\omega$. Moreover, we have $\|\mathcal{T}(t)\| = \|T\|$, for $t \in \mathbb{R}$ and $\mathcal{T}(0) = T$. The transformation $\mathcal{T}$ is continuous from $\mathbb{R}$ into W_ω. For $n \in \mathbb{N}$, $\eta \in \mathbb{R}$, $x \in \mathbb{R}$, set $g_n(\eta) := (1 - |\frac{\eta}{n}|)\chi_{[-n,n]}(\eta)$ and $\gamma_n(x) = \frac{1-\cos(nx)}{\pi x^2 n}$. We have $\widehat{\gamma_n}(\eta) = g_n(\eta)$, $\forall \eta \in \mathbb{R}$, $\forall n \in \mathbb{N}$. Clearly, $\|\gamma_n\|_{L^1} = 1$ for all n and $\lim\limits_{n\to+\infty}\int\limits_{|x|\geqq a}\gamma_n(x)dx = 0$ for $a > 0$. Set $Y_n := (\mathcal{T} * \gamma_n)(0)$. Then for $f \in L^2_\omega(\mathbb{R}^+)$ we obtain

$$\lim_{n\to+\infty}\|Y_nf - Tf\|_\omega = 0.$$

Hence, for $n \in \mathbb{N}$ and $f \in L^2_\omega(\mathbb{R}^+)$, we have

$$\begin{aligned}\|Y_n f\|^2_\omega &= \|(\mathcal{T} * \gamma_n)(0) f\|^2_\omega = \int_0^{+\infty} \left| \int_{-\infty}^{+\infty} (\mathcal{T}(y) f)(x) \gamma_n(-y) dy \right|^2 \omega(x)^2 dx \\ &\leqq \int_0^{+\infty} \left(\int_{-\infty}^{+\infty} |(\mathcal{T}(y) f)(x)| \gamma_n(-y) dy \right)^2 \omega(x)^2 dx.\end{aligned}$$

It follows from Jensen's inequality and Fubini's theorem that we have

$$\begin{aligned}\|Y_n f\|^2_\omega &\leqq \int_{-\infty}^{+\infty} \int_0^{+\infty} |(\mathcal{T}(y) f)(x)|^2 \gamma_n(-y) \omega(x)^2 dx dy \\ &\leqq \int_{-\infty}^{+\infty} \|\mathcal{T}(y)\|^2 \|f\|^2_\omega \, \gamma_n(y) dy \leqq \int_{-\infty}^{+\infty} \|T\|^2 \|f\|^2_\omega \, \gamma_n(y) dy \\ &= \|T\|^2 \|f\|^2_\omega, \ \ \forall n \in \mathbb{N}, \ \ \forall f \in L^2_\omega(\mathbb{R}^+).\end{aligned}$$

We conclude that $\|Y_n\| \leqq \|T\|$. Now consider the distribution associated to Y_n. Let K be a compact subset of $\mathbb{R}$ and let $z_K \geqq 1$ be such that $K \subset]-\infty, z_K[$. By applying the argument of the proof of Theorem 2 and Sobolev's lemma, we have for $f \in C^\infty_K(\mathbb{R})$

$$\begin{aligned}&|(TS_{z_K}(\tilde{f} g_n))(z_K)| \\ &\leqq C(K)\|T\| \left(\left(\int_{|y - z_K| \leqq M} |S_{z_K}(\tilde{f} g_n)(y)|^2 dy \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left(\int_{|y - z_K| \leqq M} |S_{z_K}(\tilde{f} g_n)'(y)|^2 dy \right)^{\frac{1}{2}} \right) \\ &\leqq C(K)\|T\| \left(\left(\int_{-M}^{M} |(\tilde{f} g_n)(x)|^2 dx \right)^{\frac{1}{2}} + \left(\int_{-M}^{M} |(\tilde{f} g_n)'(x)|^2 dx \right)^{\frac{1}{2}} \right) \\ &\leqq \tilde{C}(K)(\|f\|_\infty + \|f'\|_\infty),\end{aligned}$$

where $C(K)$ and $\tilde{C}(K)$ depend only on K. Therefore

$$|(TS_z(\tilde{f} g_n))(z)| \leqq \tilde{C}(K)(\|f\|_\infty + \|f'\|_\infty), \ \forall z \geqq z_K, \ \forall f \in C^\infty_K(\mathbb{R})$$

and we conclude that $\mu_T g_n$, defined by

$$\langle \mu_T g_n, f \rangle = \lim_{z \to +\infty} (TS_z(\tilde{f} g_n))(z),$$

is a distribution. On the other hand, we have

$$\begin{aligned}(Y_n f)(y) &= \int_{\mathbb{R}} (\mathcal{T}(-s)f)(y)\gamma_n(s)ds \\ &= \int_{\mathbb{R}} e^{-isy}(T(M_{-s}f))(y)\gamma_n(s)ds \\ &= \int_{\mathbb{R}} \langle \mu_{T,x}, f(y-x)e^{-isx}\rangle \gamma_n(s)ds \\ &= \left\langle \mu_{T,x}, f(y-x)\int_{\mathbb{R}} \gamma_n(s)e^{-isx}ds \right\rangle \\ &= \langle \mu_{T,x}, f(y-x)g_n(x)\rangle \\ &= (\mu_T g_n * f)(y), \forall y \geqq 0,\ \forall f \in C_c^\infty(\mathbb{R}^+).\end{aligned}$$

Finally, we obtain

$$Y_n f = P^+(\mu_T g_n * f),\ \forall f \in C_c^\infty(\mathbb{R}^+),\ \forall n \in \mathbb{N}.$$

Since supp $\mu_T g_n \subset [-n, n]$, this completes the proof. □

Theorem 4. *Let ω be a weight on $\mathbb{R}^+$. If $T \in W_\omega$, then there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ such that*

$$\lim_{n\to+\infty} \|T_{\phi_n} f - Tf\|_\omega = 0, \forall f \in L^2_\omega(\mathbb{R}^+)$$

and

$$\|T_{\phi_n}\| \leqq \left(\sup_{0\leqq t\leqq \frac{1}{n}} \tilde{\omega}(t)\right) \|T\|, \forall n \in \mathbb{N}.$$

Proof. Let $T \in W_\omega$ be associated to a distribution μ_T with compact support. Let $(\theta_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ be a sequence such that supp $\theta_n \subset [0, \frac{1}{n}]$, $\theta_n \geqq 0$, $\lim_{n\to+\infty} \int_{x\geqq a} \theta_n(x)$ $dx = 0$ for $a > 0$ and $\|\theta_n\|_{L^1} = 1$, for $n \in \mathbb{N}$. For $f \in L^2_\omega(\mathbb{R}^+)$ we have $\lim_{n\to+\infty} \|\theta_n * f - f\|_\omega = 0$. Set $T_n f = T(\theta_n * f)$, $\forall f \in L^2_\omega(\mathbb{R}^+)$. We conclude that $(T_n)_{n\in\mathbb{N}}$ converges to T with respect to the strong operator topology and $T_n = T_{\phi_n}$, where $\phi_n = \mu_T * \theta_n \in C_c^\infty(\mathbb{R})$. For $f \in L^2_\omega(\mathbb{R}^+)$, we have

$$\begin{aligned}\|T_n f\|^2_\omega &= \|P^+(\mu_T * \theta_n * f)\|^2_\omega = \|P^+(\theta_n * \mu_T * f)\|^2_\omega \\ &= \int_0^{+\infty} \left| \int_{\mathbb{R}} \theta_n(y)(S_y(\mu_T * f))(x)dy \right|^2 \omega(x)^2 dx \\ &\leqq \int_0^{+\infty} \int_{\mathbb{R}} \theta_n(y)|(S_y(\mu_T * f))(x)|^2 \omega(x)^2 dy dx.\end{aligned}$$

By Fubini's theorem we obtain

$$\begin{aligned}\|T_n f\|_\omega^2 &\leqq \int_0^{\frac{1}{n}} \theta_n(y) \left(\int_0^{+\infty} |(\mu_T * S_y f)(x)|^2 \omega(x)^2 dx \right) dy \\ &\leqq \int_0^{\frac{1}{n}} \theta_n(y) \|T(S_y f)\|_\omega^2 \, dy \;\leqq\; \int_0^{\frac{1}{n}} \theta_n(y) \|T\|^2 \tilde{\omega}(y)^2 \|f\|_\omega^2 dy \\ &\leqq \|T\|^2 \left(\sup_{0 \leqq y \leqq \frac{1}{n}} \tilde{\omega}(y)^2 \right) \|f\|_\omega^2.\end{aligned}$$

We deduce that $\|T_n\| \leqq (\sup_{0 \leqq y \leqq \frac{1}{n}} \tilde{\omega}(y)) \|T\|$ and Theorem 4 follows immediately from an application of Theorem 3. □

4. Representation of Wiener-Hopf operators. Set $\omega^*(x) = \omega(-x)^{-1}$, for all $x \in \mathbb{R}^-$. We introduce the space

$$L^2_{\omega^*}(\mathbb{R}^-) := \left\{ f \text{ measurable on } \mathbb{R}^- \mid \int_{\mathbb{R}^-} |f(x)|^2 \omega^*(x)^2 dx < +\infty \right\}.$$

We will consider $L^2_{\omega^*}(\mathbb{R}^-)$ as a subspace of $L^2_{\omega^*}(\mathbb{R}^-) \oplus L^2(\mathbb{R}^+)$ by setting $f(t) = 0$, for $t > 0$, when $f \in L^2_{\omega^*}(\mathbb{R}^-)$. Set

$$[f, g] := [f, g]_\omega = \int_{\mathbb{R}^+} f(x) \overline{g}(-x) dx, \ \forall f \in L^2_\omega(\mathbb{R}^+), \ \forall g \in L^2_{\omega^*}(\mathbb{R}^-).$$

We will denote by S_{a,ω^*} the translation operator from $L^2_{\omega^*}(\mathbb{R}^-) \oplus L^2(\mathbb{R}^+)$ to $L^2_{\omega^*}(\mathbb{R}^-) \oplus L^2(\mathbb{R}^+)$ defined by

$$(S_{a,\omega^*} f)(x) = f(x - a),$$

for $a \in \mathbb{R}$, $x \in \mathbb{R}$. Denote by $P^- : L^2_{\omega^*}(\mathbb{R}^-) \oplus L^2(\mathbb{R}^+) \longrightarrow L^2_{\omega^*}(\mathbb{R}^-)$ the operator defined by $P^- f = \chi_{\mathbb{R}^-} f$.

Lemma 2. *Let ω be a continuous weight on $\mathbb{R}^+$. Then*

1) *For $\alpha \in B^-_\omega := \{z \in \mathbb{C} \mid \ln R^-_\omega \leqq \operatorname{Im} z$ and $\lim_{n \to +\infty} \sum_{k=0}^{n} e^{-2k \operatorname{Im} z} \omega(k)^2 = +\infty\}$ there exists a sequence $(u_{\alpha,k})_{k \in \mathbb{N}} \subset L^2_\omega(\mathbb{R}^+)$ such that*

(4.1) i) $\|u_{\alpha,k}\|_\omega = 1, \ \ \forall k \in \mathbb{N}.$

(4.2) ii) $\lim_{k\to+\infty} \|P^+ S_{t,\omega} u_{\alpha,k} - e^{-it\alpha} u_{\alpha,k}\|_\omega = 0, \ \forall t \in \mathbb{R}.$

2) *For* $\alpha \in B^+_\omega := \{z \in \mathbb{C} \mid \operatorname{Im} z \leqq \ln R^+_\omega$ *and* $\lim_{n\to+\infty} \sum_{k=0}^{n} \frac{e^{2k\operatorname{Im} z}}{\omega(k)^2} = +\infty\}$ *there exists a sequence* $(v_{\alpha,k})_{k\in\mathbb{N}} \subset L^2_{\omega^*}(\mathbb{R}^-)$ *such that*

(4.3) i) $\| v_{\alpha,k} \|_{\omega^*} = 1, \ \ \forall k \in \mathbb{N}.$

(4.4) ii) $\lim_{k\to+\infty} \|P^- S_{t,\omega^*} v_{\alpha,k} - e^{-it\alpha} v_{\alpha,k}\|_{\omega^*} = 0, \ \ \forall t \in \mathbb{R}.$

Proof. The proof uses the same arguments as those in Section 3 in [5] (see Lemmas 4, 5, 6 ,7). Setting $f_\epsilon = \chi_{[0,\epsilon]}$ and $g_n = \sum_{p=0}^{n} e^{i(p+1)\alpha} S_p f_\epsilon$, we have just to repeat with minor modifications the argument in [5] and for this reason we omit the details. □

For $T \in B(L^2_\omega(\mathbb{R}^+))$ denote by T^* the operator in $B(L^2_{\omega^*}(\mathbb{R}^-))$ such that

$$[Tf, g] = [f, T^* g],$$

for all $f \in L^2_\omega(\mathbb{R}^+)$, $g \in L^2_{\omega^*}(\mathbb{R}^-)$.

Lemma 3. *Let* ω *be a continuous weight on* $\mathbb{R}^+$. *Then*

1) *For* $\alpha \in B^-_\omega$, *there exists a sequence* $(u_{\alpha,k})_{k\in\mathbb{N}} \subset L^2_\omega(\mathbb{R}^+)$ *such that*

$$\|u_{\alpha,k}\|_\omega = 1, \ \forall k \in \mathbb{N},$$

(4.5) $$\lim_{k\to+\infty} \| T_\phi u_{\alpha,k} - \hat{\phi}(\alpha) u_{\alpha,k} \|_\omega = 0, \ \forall \phi \in C^\infty_c(\mathbb{R}).$$

2) *For* $\alpha \in B^+_\omega$, *there exists a sequence* $(v_{\alpha,k})_{k\in\mathbb{N}} \subset L^2_{\omega^*}(\mathbb{R}^-)$ *such that*

$$\|v_{\alpha,k}\|_{\omega^*} = 1, \ \forall k \in \mathbb{N},$$

(4.6) $$\lim_{k\to+\infty} \|T^*_\phi v_{\alpha,k} - \hat{\phi}(\alpha) v_{\alpha,k}\|_{\omega^*} = 0, \ \forall \phi \in C^\infty_c(\mathbb{R}).$$

Proof. Let $\alpha \in B^-_\omega$ and let $\phi \in C^\infty_{[-a,a]}(\mathbb{R})$. Choose a sequence $(u_{\alpha,k})_{k\in\mathbb{N}} \subset L^2_\omega(\mathbb{R}^+)$ with the properties (4.1) and (4.2). We obtain

$$\begin{aligned} &\| T_\phi u_{\alpha,k} - \hat{\phi}(\alpha) u_{\alpha,k} \|^2_\omega \\ &= \int_0^{+\infty} \left| \int_{-a}^{a} \phi(y)(S_y u_{\alpha,k}(x) - e^{-iy\alpha} u_{\alpha,k}(x)) dy \right|^2 \omega(x)^2 dx \\ &\leqq \int_0^{+\infty} \|\phi\|^2_\infty \left(\int_{-a}^{a} \left| S_y u_{\alpha,k}(x) - e^{-iy\alpha} u_{\alpha,k}(x) \right| dy \right)^2 \omega(x)^2 dx, \ \ \forall k \in \mathbb{N}. \end{aligned}$$

It follows from Jensen's inequality and Fubini's theorem that we have

$$\begin{aligned}
&\|T_\phi\, u_{\alpha,k} - \hat{\phi}(\alpha) u_{\alpha,k}\|_\omega^2 \\
&\quad \leqq \|\phi\|_\infty^2 \int_{-a}^{a} \left(\int_0^{+\infty} \left| S_y\, u_{\alpha,k}(x) \;-\; e^{-iy\alpha} u_{\alpha,k}(x) \right|^2 \omega(x)^2 dx \right) dy \\
&\quad \leqq \|\phi\|_\infty^2 \int_{-a}^{a} \|P^+ S_y\, u_{\alpha,k} \;-\; e^{-iy\alpha} u_{\alpha,k}\|_\omega^2 \; dy, \;\; \forall k \in \mathbb{N}.
\end{aligned}$$

Since for $k \in \mathbb{N}$ and $y \in [-a, a]$,

$$\|P^+ S_y\, u_{\alpha,k} \;-\; e^{-iy\alpha} u_{\alpha,k}\|_\omega \;\leqq\; \sup_{s \in [-a,a]} (\tilde{\omega}(s) + |e^{-is\alpha}|) < +\infty.$$

Applying the dominated convergence theorem, we get

$$\lim_{k \to +\infty} \|T_\phi\, u_{\alpha,k} - \hat{\phi}(\alpha) u_{\alpha,k}\|_\omega = 0.$$

In the same way, by using Lemma 2, we obtain the second assertion. □

Lemma 4. *Let ω be a continuous weight on $\mathbb{R}^+$ and let $\phi \in C_c^\infty(\mathbb{R})$. Then we have*

$$|\hat{\phi}(\alpha)| \;\leqq\; \| \, T_\phi \, \|, \; \forall \alpha \in A_\omega. \tag{4.7}$$

Proof. Note that from Cauchy-Schwartz's inequality we obtain that for $z \in \mathbb{C}$ at least one of the series $\sum_{k=0}^{n} e^{-2k\mathrm{Im}\, z} \omega(k)^2$ and $\sum_{k=0}^{n} \frac{e^{2k\mathrm{Im}\, z}}{\omega(k)^2}$ diverges and we have $A_\omega \subset B_\omega^- \bigcup B_\omega^+$. Let $\phi \in C_c^\infty(\mathbb{R})$. Assume that $\alpha \in A_\omega \bigcap B_\omega^-$. Let $(u_{\alpha,k})_{k \in \mathbb{N}} \subset L_\omega^2(\mathbb{R}^+)$ be a sequence satisfying (4.5). Since $\|u_{\alpha,k}\|_\omega = 1$, for all $k \in \mathbb{N}$, we have

$$\hat{\phi}(\alpha) = \langle \hat{\phi}(\alpha) u_{\alpha,k} - T_\phi\, u_{\alpha,k} \;,\; u_{\alpha,k} \rangle + \langle T_\phi\, u_{\alpha,k} \;,\; u_{\alpha,k} \rangle, \;\; \forall k \in \mathbb{N}$$

and we obtain

$$|\hat{\phi}(\alpha)| \;\leqq |\langle \hat{\phi}(\alpha) u_{\alpha,k} - T_\phi\, u_{\alpha,k} \;,\; u_{\alpha,k} \rangle| + \|T_\phi\|, \;\; \forall k \in \mathbb{N}.$$

We have

$$\lim_{k \to +\infty} |\langle \hat{\phi}(\alpha) u_{\alpha,k} - T_\phi\, u_{\alpha,k} \;,\; u_{\alpha,k} \rangle| \;\leqq\; \lim_{k \to +\infty} \|\hat{\phi}(\alpha) u_{\alpha,k} - T_\phi\, u_{\alpha,k}\|_\omega = 0$$

and we conclude that

$$|\hat{\phi}(\alpha)| \;\leqq \|T_\phi\|.$$

If $\alpha \in A_\omega \bigcap B_\omega^+$, by using the same argument and the property (4.6), we have

$$|\hat{\phi}(\alpha)| \;\leqq \|T_\phi^*\|.$$

Taking into account the equality $\|T_\phi\| = \|T^*_\phi\|$, we obtain

$$|\hat{\phi}(\alpha)| \leqq \|T_\phi\|, \quad \forall \alpha \in A_\omega$$

and the proof is complete. □

Now we will prove our main result.

Proof of Theorem 1. Assume that ω is continuous and let $T \in W_\omega$. Let $(\phi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ be a sequence such that $(T_{\phi_n})_{n\in\mathbb{N}}$ converges to T with respect to the strong operator topology and such that $\|T_{\phi_n}\| \leqq k_n\|T\|$, where $k_n = \sup_{0\leqq y\leqq \frac{1}{n}} \tilde{\omega}(y)$ (see Theorem 4). Fix $a \in I_\omega$. We have

$$|(\widehat{\phi_n})_a(x)| = |\widehat{\phi_n}(x + ia)| \leqq \|T_{\phi_n}\| \leqq k_n\|T\|,$$

for all $x \in \mathbb{R}$. We can extract from $((\widehat{\phi_n})_a)_{n\in\mathbb{N}}$ a subsequence which converges with respect to the weak topology $\sigma(L^\infty(\mathbb{R}), L^1(\mathbb{R}))$ to a function $\nu_a \in L^\infty(\mathbb{R})$. For simplicity this subsequence will be denoted also by $((\widehat{\phi_n})_a)_{n\in\mathbb{N}}$. We have $\|\nu_a\|_\infty \leqq \lim_{n\to+\infty} (\sup_{0\leqq t\leqq \frac{1}{n}} \tilde{\omega}(t))\|T\|$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \left((\widehat{\phi_n})_a(x) - \nu_a(x)\right) g(x)\, dx = 0, \quad \forall g \in L^1(\mathbb{R}).$$

Notice that

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \left((\widehat{\phi_n})_a(x)\widehat{(f)}_a(x) - \nu_a(x)\widehat{(f)}_a(x)\right) g(x)\, dx = 0,$$

$$\forall g \in L^2(\mathbb{R}),\ \forall f \in C_c^\infty(\mathbb{R}).$$

We conclude that, for $f \in C_c^\infty(\mathbb{R})$, $((\widehat{\phi_n})_a\widehat{(f)}_a)_{n\in\mathbb{N}}$ converges with respect to the weak topology of $L^2(\mathbb{R})$ to $\nu_a\widehat{(f)}_a$. Since we have $(T_{\phi_n}f)_a = P^+((\phi_n)_a * (f)_a) = P^+\mathcal{F}^{-1}\,((\widehat{\phi_n})_a\widehat{(f)}_a)$, the sequence $((T_{\phi_n}f)_a)_{n\in\mathbb{N}}$ converges with respect to the weak topology of $L^2(\mathbb{R})$ to $P^+\mathcal{F}^{-1}(\nu_a\widehat{(f)}_a)$. Moreover, we have

$$\int_0^{+\infty} |(T_{\phi_n}f)_a(x) - (Tf)_a(x)||g(x)|dx$$

$$\leqq C_{a,g}\|T_{\phi_n}f - Tf\|_\omega\ ,\ \forall g \in C_c^\infty(\mathbb{R}),$$

where $C_{a,g} > 0$ depends only on g and a. Then, we obtain that $((T_{\phi_n}f)_a)_{n\in\mathbb{N}}$ converges in the sense of distributions to $(Tf)_a$. Thus, we conclude that $(Tf)_a = P^+\mathcal{F}^{-1}(\nu_a\widehat{(f)_a})$ and $(Tf)_a \in L^2(\mathbb{R}^+)$.

Below, we assume that $\overset{\circ}{I}_\omega \neq \emptyset$. Since $(\widehat{\phi_n})_{n\in\mathbb{N}}$ is a uniformly bounded sequence of holomorphic functions on $\overset{\circ}{A}_\omega$, we can replace $(\widehat{\phi_n})_{n\in\mathbb{N}}$ by a subsequence which converges to a function $\nu \in \mathcal{H}(\overset{\circ}{A}_\omega)$ uniformly on every compact set. Thus, for all $a \in I_\omega$, the sequence $(\widehat{\phi_n}(.+ia))_{n\in\mathbb{N}}$ converges to $\nu(.+ia)$ in the sense of distributions. On the other hand, the sequence $(\widehat{(\phi_n)_a})_{n\in\mathbb{N}}$ converges to ν_a with respect to the topology $\sigma(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$, and we deduce that

$$\nu(x+ia) = \nu_a(x),\ a.e. \text{ for } a \in \overset{\circ}{I}_\omega .$$

It is clear that $\|\nu\|_\infty \leqq \lim_{n\to+\infty} (\sup_{0\leqq t\leqq \frac{1}{n}} \tilde{\omega}(t))\|T\|$. If ω is such that $\lim_{n\to+\infty} \sup_{0\leqq t\leqq \frac{1}{n}} \tilde{\omega}(t) = 1$, we obtain $\|\nu\|_\infty \leqq \|T\|$. If we don't assume that ω is continuous we have $\|\nu\|_\infty \leqq C_\omega\|T\|$, where C_ω is the constant defined in the introduction. To obtain this we apply the equivalence of ω to a special continuous weight ω_0 (see Section 1). This completes the proof. □

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